XXXII High School Math Contest

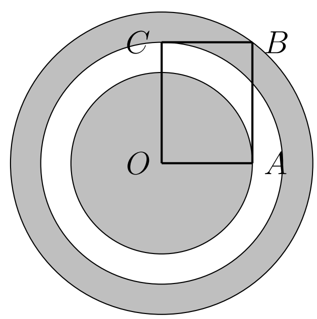
University of South Carolina

February 3rd, 2018

Problem 1. For what values of the real number a, does the quadratic equation $x^2+ax+a=0$ have two real zeros, with one of these zeros positive and the other zero negative?

(a) $a < 0$ (b) $0 < a < 4$ (c) $a > 1$ (d) $a > 4$ (e) no such values exist

Problem 2. Given a rectangle $OABC$ as in the figure below, with $OA = a$ and $OC = c$, three concentric circles with radii $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ are drawn. Compute the difference between the area of the inner circle with radius $\overline{OA}$ and the area of the annulus generated by the concentric circles with radii $\overline{OB}$ and $\overline{OC}$. (An annulus generated by two concentric circles is the region between the circles.)



(a) 0 (b) $\pi a^2 - \frac{\pi}{2}c^2$ (c) $\frac{\pi}{2}(a-c)^2$ (d) $\frac{\pi}{3}(a-c)^2$ (e) $\pi c^2 - \frac{\pi}{2}a^2$

Problem 3. How many real solutions does the following equation have?

$$x^2 + \sqrt{x-2} = 4 + \sqrt{4-2x}$$

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

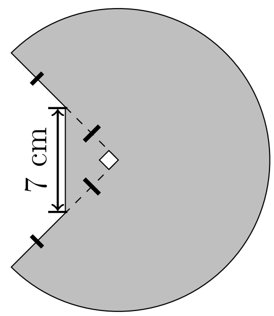
Problem 4. Let $x = \log_{10}(81)$ and $y = \log_{10}(25)$. If one expresses $\log_{10}(6)$ as $ax + by + c$ where a, b, and c are rational numbers, then what is $a + b + c$?

(a) $-\frac{7}{4}$ (b) $-\frac{5}{6}$ (c) $\frac{3}{4}$ (d) $\frac{5}{6}$ (e) $\frac{7}{4}$

Problem 5. If x is a real number such that $x - x^{-1} = 3$, what is the value of $x^3 - x^{-3}$?

(a) 11 (b) $8 + 5\sqrt{13}$ (c) 27 (d) $5 + 8\sqrt{13}$ (e) 36

Problem 6. The shaded region below is a union of three quarters of a circle and a right triangle whose vertices are the center of the circle and the two midpoints of the the shown radii. Find its area.



(a) $\frac{147\pi}{2} + \frac{49}{4}$ (b) $147\pi + \frac{49}{4}$ (c) $147\pi + \frac{49}{2}$ (d) $196\pi + \frac{49}{4}$ (e) $196\pi + \frac{49}{2}$

Problem 7. Consider all integers a such that both zeros of the following quadratic equation are integers.

$$x^2 - ax + 2a = 0$$

What is the sum of all such integers a?

(a) 7 (b) 8 (c) 10 (d) 12 (e) 16

Problem 8. Let $a, b, c > 1$ and $x > 1$ so that $\log_{ab}(x) = 9$, $\log_{bc}(x) = 18$ and $\log_{abc}(x) = 8$. Determine $\log_{ac}(x)$.

(a) 9 (b) 12 (c) 18 (d) 24 (e) 36

Problem 9. The x– and y–coordinates of two different points $(\sqrt{\pi}, a)$ and $(\sqrt{\pi}, b)$ satisfy the following equation:

$$y^2 + x^4 = 2x^2y + 1.$$

What is $|a - b|$?

(a) 1 (b) $\frac{\pi}{2}$ (c) 2 (d) $\sqrt{1 + \pi}$ (e) $1 + \sqrt{\pi}$

Problem 10. Let S be a sphere of radius 8 and let C be a cube whose eight vertices lie on the surface of S. What is the volume of C?

(a) $\frac{512\sqrt{3}}{3}$ (b) $\frac{512\sqrt{6}}{3}$ (c) $\frac{1024\sqrt{3}}{3}$ (d) $\frac{4096\sqrt{3}}{9}$ (e) $1024\sqrt{2}$

Problem 11. A person rolls four fair six-sided dice. What is the probability that the person rolls exactly one 1 and exactly one 2?

(a) $\frac{4}{27}$ (b) $\frac{16}{81}$ (c) $\frac{302}{6^4}$ (d) $\frac{500}{6^4}$ (e) $\frac{65}{81}$

Problem 12. Suppose $x \neq y$ and two sequences of numbers

$$x, a_1, a_2, a_3, y, \qquad \text{and} \qquad b_1, x, b_2, b_3, y, b_4$$

are both arithmetic sequences. What is the value of $\dfrac{b_4 - b_3}{a_2 - a_1}$?

(a) 2/3 (b) 4/3 (c) 5/3 (d) 7/3 (e) 8/3

Problem 13. If the function $f(x)$ is defined on $\mathbb{R}$ and satisfies both $f(10+x) = f(10-x)$ and $f(20-x) = -f(20+x)$, then $f(x)$ is

(a) periodic, but neither even nor odd (b) even and periodic (c) even, but not periodic (d) odd, but not periodic (e) odd and periodic

Problem 14. How many solutions of the equation $\cos(7x) = \cos(5x)$ are in the interval $[0, \pi]$?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Problem 15. What is the radius of the inscribed circle in a triangle whose sides are 12, 13, and 5?

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2 (e) $\sqrt{5}$

Problem 16. What is the largest power of 2 that divides $13^4 - 11^4$?

(a) 8 (b) 16 (c) 32 (d) 64 (e) 128

Problem 17. Let $F_0 = 0$ and $F_1 = 1$. For all $n \geq 2$, define F_n to be the remainder of $F_{n-1} + F_{n-2}$ divided by 3. The sequence starts as follows:

$$0, 1, 1, 2, 0, 2, \ldots$$

What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$?

(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

Problem 18. Consider the set of all fractions x/y where x and y are relatively prime positive integers. How many of these fractions have the property that if both the numerator and the denominator are increased by 1, the value of the fraction is increased by 10%?

(a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many.

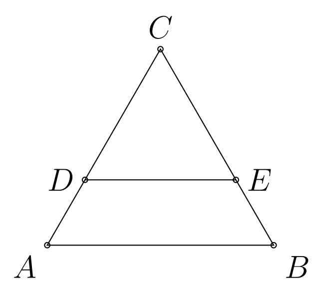
Problem 19. Tom has 12 coins, each of which is a nickel or a dime. There are exactly 17 different values that can be obtained as combinations of one or more of his coins. How many dimes does Tom have?

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

Problem 20. A regular 6-sided die with the numbers 1, 2, 3, 4, 5 and 6 is rolled twice. Next, another regular 6-sided die is rolled two times. What is the probability that the sum of the numbers rolled on the first 6-sided die is greater than the sum of the numbers rolled on the second 6-sided die?

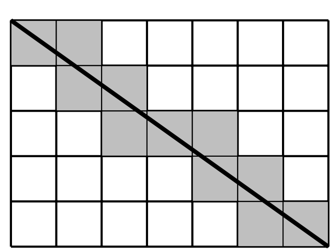
(a) $\frac{73}{648}$ (b) $\frac{77}{648}$ (c) $\frac{77}{432}$ (d) $\frac{575}{1296}$ (e) $\frac{1}{2}$

Problem 21. In the equilateral triangle $\triangle ABC$, the point D is on $\overline{AC}$, the point E is on $\overline{BC}$, and $\overline{DE}$ is parallel to $\overline{AB}$. If the perimeter of the triangle $\triangle DEC$ is equal to the perimeter of the trapezoid $ABED$, what is the ratio of the areas of the triangle $\triangle DEC$ and the trapezoid $ABED$?



(a) $\frac{1}{3}$ (b) $\frac{9}{16}$ (c) 1 (d) $\frac{9}{7}$ (e) $\frac{16}{9}$

Problem 22. A diagonal of this 5×7 rectangle passes through 11 squares. (We say that a line passes through a square if the square and the line have at least two common points.)



How many squares will the diagonal of a 2016×2018 rectangle pass through?

(a) 4032 (b) 4033 (c) 4034 (d) 4035 (e) 4036

Problem 23. Suppose s and t are integers satisfying $s\sqrt{s-t} = t\sqrt{2t+s}$. If $5 \le s \le 20$ and $5 \le t \le 20$, then how many such solutions are there?

(a) 1 (b) 4 (c) 5 (d) 6 (e) 16

Problem 24. For each positive integer n, the parabola $y = (n^2+n)x^2-(2n+1)x+1$ intersects the x–axis at the points A_n and B_n. What is the value of $A_1B_1 + A_2B_2 + \cdots + A_{2018}B_{2018}$?

(a) $\frac{2017}{2019}$ (b) $\frac{2016}{2017}$ (c) $\frac{2017}{2018}$ (d) $\frac{2018}{2019}$ (e) $\frac{2018}{2017}$

Problem 25. For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct zeros, and each zero of $g(x)$ is also a zero of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

(a) -9009 (b) -8008 (c) -7007 (d) -6006 (e) -5005

Problem 26. Suppose x, y and z are three real numbers such that $3x$, $4y$, $5z$ is a geometric sequence and $1/x$, $1/y$, $1/z$ is an arithmetic sequence. What is the value of $(x/z) + (z/x)$?

(a) 32/15 (b) 34/15 (c) 37/15 (d) 38/15 (e) 64/15

Problem 27. If a_1 and a_2 are positive integers with $a_1 > a_2$ we define a sequence of integers as follows: $a_{n+2} = |a_{n+1} - a_n|$ for $n = 1, 2, \ldots$ if $a_n > 0$ and $a_{n+1} > 0$, and we stop when $a_{n+1} = 0$. For example, if $a_1 = 10$ and $a_2 = 6$, we get the sequence $10, 6, 4, 2, 2, 0$.
If we start with positive integers a_1 and a_2, both between 1 and 10, what is the length of the longest sequence one can obtain? (The length of a sequence is the number of terms in the sequence, for example the sequence $10, 6, 4, 2, 2, 0$ has length 6).

(a) 6 (b) 7 (c) 9 (d) 10 (e) 17

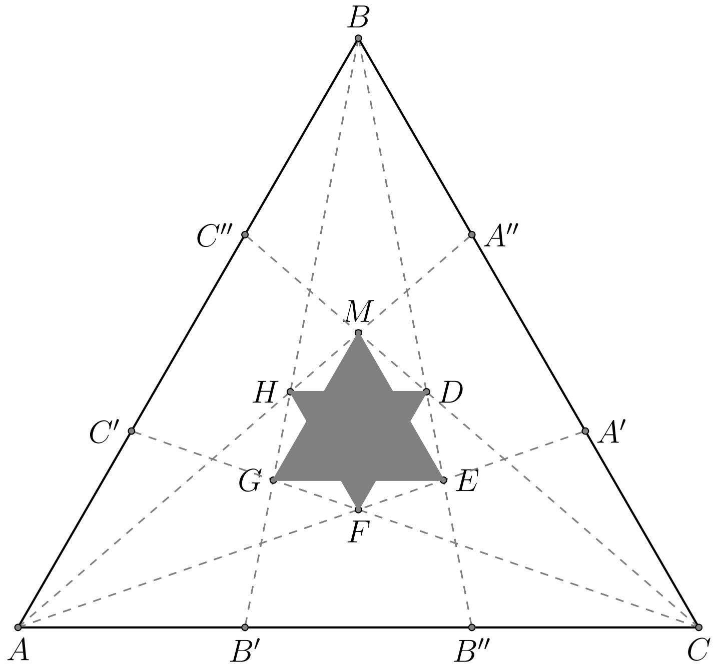
Problem 28. Many high schools in South Carolina participated in a math contest. Each such high school sent in a team of three contestants. Suppose that each contestant earned a different score. The contestant that scored the median among all students had the highest score on their team while their teammates placed 59^{th} and 106^{th}. How many schools participated in the contest?

(a) 36 (b) 37 (c) 38 (d) 39 (e) There is not enough information to determine the number

Problem 29. How many different ways can one order the integers 1 through 5 so that no three consecutive integers in the ordering are in increasing order?

(a) 60 (b) 64 (c) 70 (d) 72 (e) 120

Problem 30. Given an equilateral triangle $\triangle ABC$, consider the points A' and A'' that trisect side $\overline{BC}$, B' and B'' that trisect side $\overline{AC}$, and C' and C'' that trisect side $\overline{AB}$ as shown. What percentage of the area of $\triangle ABC$ is the area of the shaded star?



(a) 6% (b) 7% (c) 8% (d) 9% (e) 10%